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Dynamical properties of PWD0L systems

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Abstract

The definitions of a piecewise deterministic zero Lindenmayer (PWD0L) scheme and system are given, and dynamical properties of such systems are introduced. Harrison (1994) showed that given an arbitrary finite alphabet A, the emptiness problem is undecidable for the class of languages which are intersections of a D0L language and a context-sensitive language. This result is used to prove that many dynamical properties of PWD0L systems(such as finiteness, periodicity, etc) are in general undecidable over a two-member context-sensitive partition of A^*.

The idea of a morphic equivalence relation on A^* (A is finite) is defined and the idea of a finite morphic refinement is introduced. Harrison (1994) showed that every regular language and its complement is refined by a finite partition which is induced by a morphic congruence. Using this theorem, it is shown that dynamical properties such as finiteness and periodicity are in general decidable for RWD0L systems (i.e. a PWD0L system over a finite partition of A^* made up of all regular languages).

1. Introduction

Intuitively, the notion of a *PWD0L scheme* is very simple, and it is a natural extension of a D0L scheme. A PWD0L scheme is an ordered triple made up of a finite alphabet, a finite partition of A^*, and a rewriting function called a piecewise endomorphism of A^*, in which the rewriting of a word depends upon which member of the finite partition the word is in. Given this definition, analogous dynamical properties to those which have been studied for D0L schemes (and systems) can be made. These concepts will become more clear following the definitions given below.

Definition 1.1. A *PWD0L scheme* S is an ordered triple $S = (A, \mathbf{P}, H)$ where
(1) A is a finite alphabet.
(2) $\mathbf{P} = \{P_1, \ldots, P_k\}$ is a finite partition of A^*.

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(3) $H = (h_1, \ldots, h_k)$ is called a *piecewise endomorphism* of A^*, i.e. $(w)H = (w)h_i$ if $w \in P_i$ for $1 \leqslant i \leqslant k$ and each h_i is an endomorphism of A^*.

A PWD0L scheme in which every member of the finite partition is a regular language is called an *RWD0L scheme.*

Definition 1.2. A *PWD0L(RWD0L) system* is a 4-tuple $S_w = (A, \mathbf{P}, H, w)$ in which $(A, \mathbf{P}, H)$ is a PWD0L(RWD0L) scheme and w is called the *initial word* of S_w. The *language generated by* S_w, denoted $(L(S_w))$, is defined by $L(S_w) = \{(w)H^i \text{ such that } i \geqslant 0\}$.

Definition 1.3. Let A be a finite alphabet. An *omega word* Ω is a sequence of elements from A.

Definition 1.4. Let S_w be a PWD0L system with partition $\mathbf{P}$ in which $|\mathbf{P}| = k$ for some $k \geqslant 1$. Then the *omega word generated by* S_w (denoted Ω_{S_w}) is the following omega word defined over $\{1, \ldots, k\}$ as follows:

$$(\Omega_{S_w})_i = j \in \{1, \ldots, k\} \text{ such that } (w)H^{i-1} \in P_j \text{ for } i \geqslant 1.$$

The *control word of length i generated by* S_w is the prefix of length i of Ω_{S_w}.

Definition 1.5. *A PWD0L system* S_w is *finite* if there exists $i \geqslant 0$ and $p \geqslant 1$ such that $(w)H^{i+p} = (w)H^i$. (Hence $L(S_w)$ is a finite set.)

Definition 1.6. A PWD0L system S_w is *periodic* if there exists $i \geqslant 0$ and $p \geqslant 1$ such that

$$(w)H^{i+kp} = ((w)H^i)^{e_k}$$

for every $k \geqslant 1$, where each $e_k \geqslant 0$.

Definition 1.7. A PWD0L system S_w is called *power repetitive* if there exists $i \geqslant 0$, $p \geqslant 1$, and $e \geqslant 0$ such that $(w)H^{i+p} = ((w)H^i)^e$.

The following definition is an analog of one made by Ginsburg [2]. The word bounded will be capitalized as to not confuse it with a D0L system being bounded by some growth function will respect to the lengths of its iterates.

Definition 1.8. A PWD0L system S_w is called *BOUNDED* if there exists $n \geqslant 0$ and words $x_1, \ldots, x_n \in A^*$ such that $L(S_w) \subseteq x_1^* \ldots x_n^*$.

The following definition is an analog of one made by Linna for D0L systems, which describes PWD0L systems which generate prefix codes.

Definition 1.9. A PWD0L system S_w is called *nonprefix* if there exists $i \geqslant 0$, $p \geqslant 1$ and $s \in A^+$ such that $(w)H^{i+p} = (w)H^i s$. Otherwise, S_w is called *prefix*.

For Definitions 1.5–1.7 and 1.9, the smallest i satisfying each definition is called the *index* of S_w. Finally, given the index, the smallest p satisfying each definition is called the period of S_w.

Definition 1.10. Define for a PWD0L scheme S, any $i \geqslant 0$ and $p \geqslant 1$,
(i) $Fin_{(S,i,p)} = \{w \in A^* \text{ such that } S_w \text{ is finite with index } i \text{ and period } p\}$.
(ii) $Fin_{(S,i)} = \bigcup_{p=1}^{\infty} Fin_{(S,i,p)}$.
(iii) $Fin_S = \bigcup_{i=0}^{\infty} Fin_{(S,i)}$.

Also, there are analogous definitions for the power repetitive, periodic and nonprefix words of a PWD0L scheme S. Note also that $Fin_S = \{w \in A^* \mid S_w \text{ is finite}\}$ and that corresponding definitions for power repetitive, periodic and nonprefix can be made as in the finite case. Finally, $BOUNDED_S = \{w \in A^* \mid S_w \text{ is bounded}\}$.

Definition 1.11. A word $v \in A^+$ is called *primitive* if when $v = w^e$ for some $w \in A^*$ and $e \geqslant 0$, then $e = 1$.

It follows that every w in A^+ has a unique primitive root [10].

Definition 1.12. Consider the *power equivalence* $\sim_{\text{power}}$ on A^* defined by $x \sim_{\text{power}} y$ if and only if
(1) $x = y = \lambda$ or
(2) x and y are both powers of the same unique primitive root $v \in A^+$.

Example 1.13. Consider $S = (A, \mathbf{P}, H)$ where
(1) $A = \{a, b\}$,
(2) $P = \{L, L^c\}$ where $L = (ab)^+$,
(3) $H = (h, 1_{A^*})$ where $h: A \to A^+$ is defined by $(a)h = aba$ and $(b)h = bab$.
Then $Fin_{(S,0,1)} = Fin_{(S,0)} = Fin_S = L^c$ and $Per_S = PowerRep_S = A^*$ and *NonPrefix*$_S = L$. For any $w \in L$, the omega word generated by S_w is 1^w and for any $w \in L^c$, the omega word generated by S_w is 2^w; where $h_1 = h$ and $h_2 = 1_{A^*}$.

In Section 2, the aforementioned dynamical properties of bounded index and period for PWD0L systems will be studied.

2. Dynamical properties of PWD0L systems of bounded index

Notation. Given $k \geqslant 1$, $j \in \{1, \ldots, k\}$ and $i_1, \ldots, i_j \in \{1, \ldots, k\}$, the endomorphism $h_{i_1 \ldots i_j} = h_{i_1} h_{i_2} \ldots h_{i_j}$ (Note: When $j = 0$, $h_{i_1 \ldots i_j}$ is the identity morphism on A^*.)

Proposition 2.1. *Let $S = (A, \mathbf{P}, H)$ be a PWDOL scheme over a finite partition $\mathbf{P}$ of A^*. Then,*

$$Fin_{(S,0,1)} = \bigcup_{i=1}^{k} (Fin_{(S,0,1)} \cap P_i),$$

where $k = |\mathbf{P}|$, $H = (h_1, \ldots, h_k)$ and $S_i = (A, h_i)$ is a DOL scheme for each $i \in \{1, \ldots, k\}$.

For each $p \geq 2$,

$$(*) \qquad Fin_{(S,0,p)} = \bigcup_{i_1 \ldots i_p \in \{1,\ldots,k\}^p} (Fin_{(Si_1,\ldots,i_p,0,1)} \cap P_{i_1 \ldots i_p}) \Big\backslash \bigcup_{k=1}^{p-1} Fin_{(S,0,k)},$$

where $S_{i_1 \ldots i_p} = (A, h_{i_1 \ldots i_p})$ is a DOL scheme and $P_{i_1 \ldots i_p} = \bigcap_{j=0}^{p-1} (P_{i_{j+1}}) h_{i_1 \ldots i_j}^{-1}$.

Analogous propositions exist for $PowerRep_{(S,0,p)}$ *and* $NonPrefix_{(S,0,p)}$, respectively Note as well that if $\mathbf{P}$ coarsens the power equivalence then periodicity and power repetitiveness are equivalent for S.

Proof. Assume x is in the right-hand side of $(*)$. Then $(x)h_{i_1 \ldots i_p} = x$. Since $x \in P_{i_1 \ldots i_p}$, this implies that $(x)H^p = (x)h_{i_1 \ldots i_p} = x$. Thus $L(S_x)$ (where $S_x = (A, \mathbf{P}, H, x)$) is finite with index 0 and period $\leq p$. Finally, since $x \notin \bigcup_{k=1}^{p-1} Fin_{(S,0,k)}$, the period of S_x is p and so $x \in Fin_{(S,0,p)}$.

Conversely, assume x is in the left-hand side of $(*)$. Let $i_1 \ldots i_p$ be the prefix of Ω_{S_x} of length p. Then $x \in Fin_{(S_{i_1 \ldots i_p},0,1)} \cap P_{i_1 \ldots i_p}$ and since the period of S_x is $p, x \notin \bigcup_{k=1}^{p-1} Fin_{(S,0,k)}$ as required. $\square$

The following example will show that the aforementioned proposition need not hold in general for $Per_{(S,0,1)}$.

Example 2.2. Consider $S = (A, \mathbf{P}, H)$, where $A = \{a, b\}$, $\mathbf{P} = \{L, L^c\}$ where $L = \{ab\}$ and $H = (h_1, h_2)$ is defined as follows.

$$(a)h_1 = aba, \qquad (a)h_2 = a^2, \qquad (b)h_1 = b \quad \text{and} \quad (b)h_2 = b.$$

Then, $ab \in Per_{(S_1,0,1)} \cap L_1$ ($L_1 = L$) and $S_1 = (A, h_1)$) but $ab \notin Per_{(S,0,1)}$, since $(ab)H^{i+1} = (a^{2^i} b)^2$ for each $i \geq 0$. Note that the aforementioned example is power repetitive but not periodic.

Proposition 2.3. *Lt $S = (A, \mathbf{P}, H)$ be a PWDOL scheme and let $i \geq 1$. Then,*

$(**)$

$$Fin_{(S,i,p)} = \bigcup_{j_1 \ldots j_i \in \{1,\ldots,k\}^i} ((Fin_{(S,0,p)}) h_{j_1 \ldots j_i}^{-1} \cap P_{j_1 \ldots j_i}) \Big\backslash \left(\bigcup_{l=0}^{i-1} \left(\bigcup_{q=1}^{p} Fin_{(S,l,q)} \right) \right).$$

Note that Per or PowerRep can be substituted for Fin in (**) if **P** *coarsens the partition induced by the power equivalence. Also,*

$$(***)\qquad Per_{(S,i,p)} = \bigcup_{j_1 \ldots j_i \in \{1,\ldots,k\}^i} ((Per_{(S,0,p)})h^{-1}_{j_1 \ldots j_i} \cap P_{j_1 \ldots j_i}) \Big\backslash \left(\bigcup_{l=0}^{i-1} Per_{(S,l)} \right).$$

Fin,NonPrefix and PowerRep can be substituted for Per in (***).

We will first prove the more general result stated in (***), and then prove (**) using *Per*.

Proof (***). Let x be in the left-hand side of (***). Let $j_1 \ldots j_i$ be the control word of length i generated by S_x. Then, $(x)h_{j_1 \ldots j_i} \in Per_{(S,0,p)})$. Also, since $j_1 \ldots j_i$ is the control word of length i generated by S_x, $x \in P_{j_1 \ldots j_i}$. Finally, since the index of S_x is i, $x \notin Per_{(S,l)}$ for any $l \in \{0, \ldots, i-1\}$.

Conversely, let x be in the right-hand side of (***). Since $(x)H^i = (x)h_{j_1 \ldots j_i} \in Per_{(S,,0,p)}$, the index of $S_x \leqslant i$. But now, since $x \notin \bigcup_{k=0}^{i-p} Per_{(S,k)}$, the index of $S_x = i$. Finally, since $(x)H^i \in Per_{(S,0,p)}$, the period of S_x is p, completing the proof of (***).

We will now proceed with the proof of (**). For (**), the ($\subseteq$) part is virtually identical to the proof in (***) given above and will be omitted. Thus, we will now prove ($\supseteq$) for (**).

Let x be in the right-hand side of (**). It is clear that the index of S_x is $\leqslant i$. Suppose the index m of S_x is less than i. Since $x \notin \bigcup_{l=0}^{i-1} (\bigcup_{q=1}^{p} Per_{(S,l,q)})$, the period of S_x must be some $q > p$. Now, since **P** coarsens the power equivalence and S_x is periodic with index m and period q, it follows that $\{(x)H^{m+\alpha q+r}\}_{\alpha=0}^{\infty} \subseteq ((x)H^{m+r})^*$ for $0 \leqslant r \leqslant q$. Similarly, since x is in the right-hand side of (**), it follows that $\{(x)H^{i+\beta p+s}\}_{\beta=0}^{\infty} \subseteq ((x)H^{i+s})^*$ for $0 \leqslant s < p$. Thus, there exists α, β and such that $(x)H^{m+\alpha q} = ((x)H^m)^{e_\alpha} = (x)H^{i+\beta p+s}$. But then since **P** coarsens the power equivalence and H is a piecewise endomorphism, it follows that $(((x)H^m)^{e_\alpha})H^p = ((x)H^m)^{\bar{e}e_\alpha}$ for some $\bar{e} \geqslant 0$. But since H^p is a piecewise endomorphism and the partition **P** coarsens the power equivalence, if follows that $(x)H^{m+p} = ((x)H^m)^{\bar{e}}$ and so since **P** coarsens the power equivalence, $\{(x)H^{m+\kappa p}\}_{\kappa=0}^{\infty} \subseteq ((x)H^m)^*$. Thus, the period of S_x would be $\leqslant p$ which is contradictory. Thus, the index of S_x must be i. Finally, as in the proof of (***), the period of S_x must be p and so we are done. □

A theorem which encompasses many significant results in D0L-system theory can now be applied to Propositions 2.1 and 2.3 to yield an immediate corollary.

Theorem 2.4 (Lando [8] and Linna [9]). *Let $S = (A, h)$ be a D0L scheme. Then, $Fin_{(S,i,p)}$, $Per_{(S,i,p)}$, $Fin_{(S,i)}$, $Per_{(S,i)}$, Fin_S and Per_S are constructable regular sets. Moreover, $NonPrefix_{(S,i,p)}$, $NonPrefix_{(S,i)}$ and $NonPrefix_S$ are recursive sets.*

The above theorem yields the following result as an immediate corollary.

Corollary 2.5. *Let $S = (A, \mathbf{P}, H)$ be a PWD0L scheme over a regular (CF, CS, recursive) partition of A^*. Then:*

(i) *$PowerRep_{(S,0,p)}$ is a regular (CS, CS, recursive) set. $NonPrefix_{(S,0,p)}$ is a recursive set.*

(ii) *$Fin_{(S,i,p)}$ is a regular (CS, CS, recursive) set for all $i \geqslant 0$. If $\mathbf{P}$ coarsens the partition induced by the power eqivalence, then $Per_{(S,i,p)} = PowerRep_{(S,i,p)}$ is a regular (CS, CS, recursive) set.*

Proof. (i) follows from Proposition 2.1, Theorem 2.4, and the closure properties for regular (CF, CS, recursive) languages used in the construction in Proposition 2.1. (Note: For information on standard closure properties, see [6]. For co-CS is CS, see [7].)

(ii) follows from Propositions 2.1 and 2.3, Theorem 2.4, and the closure properties for regular (CF, CS, recursive) languages used in the constructions in Propositions 2.1 and 2.3, respectively. □

The following example will show how some of the sets mentioned in the previous corollary are constructed.

Example 2.6. Define $S = (A, \mathbf{P}, H)$ as follows:

(i) $A = \{a, b, c\}$.

(ii) $\mathbf{P} = \{L_{(0,0)}, L_{(0,1)}, L_{(1,0)}, L_{(1,1)}\}$ where $L_{(i,j)} = \{w \in A^* \mid \#_a(w) \equiv i(2)$ and $\#_b(w) \equiv j(2)\}$.

(iii) Finally, define $H = (h_{(0,0)}, h_{(0,1)}, h_{(1,0)}, h_{(1,1)})$, by $(a)h_{(i,j)} = ab^i$, $(b)h_{(i,j)} = a^j b$ and $(c)h_{(i,j)} = c$ for each $(i,j) \in \{0,1\}^2$.

Then, $Fin_{(S,0,1)} = \bigcup_{(i,j)\in\{0,1\}^2} (Fin_{(S_{(i,j)},0,1)} \cap L_{(i,j)})$. But, $Fin_{(S_{(0,0)},0,1)} = A^*$, $Fin_{(S_{(0,1)},0,1)} = \{a,c\}^*$, $Fin_{(S_{(1,0)},0,1)} = \{b,c\}^*$ and finally $Fin_{(S_{(1,1)},0,1)} = \{c\}^*$. Thus, it follows that $Fin_{(S,0,1)} = L_{(0,0)}$.

Finally, $Fin_{(S,1,1)} = \bigcup_{(i,j)\in\{0,1\}^2} ((L_{(0,0)})h_{i,j}^{-1} \cap L_{(i,j)}) \backslash L_{(0,0)} = L_{(1,1)}$.

We now wish to see if bounds exist for the period for properties such as finiteness, periodicity and power repetitiveness. First, we will examine if such a bound exists for finiteness. To do this, some definitions are needed.

Definition 2.7. A piecewise endomorphism $H = (h_1, \ldots, h_k)$ is called *λ-free* if for each $i \in \{1, \ldots, k\}$ and each $l \in A$, $(l)h_i \neq \lambda$, i.e. H does not erase. H is called *piecewise injective* if for each $i \in \{1, \ldots, k\}$, h_i is injective.

This leads to a proposition about $Fin_{(S,0)}$ for a PWD0L scheme S when H is λ-free.

Proposition 2.8. *Let $S = (A, \mathbf{P}, H)$ be a PWD0L scheme in which H is λ-free. Assume $x \in Fin_{(S,0)}$. Then, there exists $p \leqslant |A|^{|A|}$ such that $x \in Fin_{(S,0,p)}$. Thus, the period for finiteness of PWD0L system in which H is λ-free is $\leqslant |A|^{|A|}$.*

Proof: Since x is finite with index 0, there exists some positive integer period p such that $x \in Fin_{(S,0,p)}$. But, since H is λ-free, $(l)H^i = \bar{l} \in A$ for each letter l of x and $i \in \{1, \ldots, p\}$, i.e. each of the iterates of H is a letter to letter function when restricted to the letters contained in x. Thus, since there are $\leqslant |A|$ distinct letters in x which can only assume at most $\leqslant |A|$ different values, the period of S_x must be $\leqslant |A|^{|A|}$ as required. □

The next step is to state a proposition about obtaining $Fin_{(S,i)}$ inductively from the finite words of lower index which will be stated without proof.

Proposition 2.9. *Let $S = (A, \mathbf{P}, H)$ be a PWD0L scheme. Then, for $i \geqslant 1$,*

$$Fin_{(S,i)} = \left[\bigcup_{j=1}^{k} (Fin_{(S,i-1)}) h_j^{-1} \cap P_j) \right] \Big\backslash \left[\bigcup_{j=0}^{i-1} Fin_{(S,j)} \right],$$

where $k = |\mathbf{P}|$. Note that analogous propositions hold for $Per_{(S,i)}$, $PowerRep_{(S,i)}$ and $NonPrefix_{(S,i)}$, respectively.

Thus, we obtain the following corollary from Propositions 2.8 and 2.9.

Corollary 2.10. *Let $S = (A, \mathbf{P}, H)$ be a PWD0L scheme in which H is λ-free and each member of $\mathbf{P}$ is regular (CF, CS, recursive). Then, for all $i \geqslant 0$, $Fin_{(S,i)}$ is regular (CS, CS, recursive) and so finiteness of index i is decidable for a PWD0L system S_x in which H is λ-free and each member of $\mathbf{P}$ is recursive (assuming $\mathbf{P}$ has been constructed).*

Proof. Follows from Propositions 2.8, 2.9 and the closure properties for regular (CF, CS, recursive) languages used in the construction in Proposition 2.9. □

In Section 3, we shall see that it is essential for H to be λ-free to obtain the desired decidability results (unless further restrictions are placed upon the members of the finite partition). The following theorem is a useful tool in obtaining a decidability result for periodicity of bounded index when the piecewise endomorphism is piecewise injective.

Theorem 2.11. *Let h be an injective endomorphism of A^*. Assume y is primitive and $(y)h = v^e$, where v is primitive. Then, $|y| \leqslant |v|$. (*)*

Proof. Suppose there exist primitives y and v and an injective morphism h such that $(y)h = v^e$ and $|y| > |v|$. Let y denote a counterexample of minimum length.

Since $|y| > |v|$, it follows from the pigeonhole principle that there exists $\bar{p}_1$ and $\bar{p}_1\bar{x} \in A^+$ with $\lambda < \bar{p}_1 < \bar{p}_1\bar{x} \leqslant y$ such that

$$(\bar{p}_1)h = v^{\varepsilon_1}p \quad \text{and} \quad (\bar{p}_1\bar{x})h = v^{\varepsilon_2}p,$$

where $0 \leqslant \varepsilon_1 < \varepsilon_2$, $v = ps$ and $y = \bar{p}_1 \bar{x} \bar{s}_1$, where p, s and $\bar{s}_1 \in A^*$. Thus, $(\bar{x})h = (sp)^{\varepsilon_2 - \varepsilon_1}$, where sp is primitive since $ps = v$ is primitive. Hence, $(\bar{s}_1)h = s(ps)^{e - \varepsilon_2 - 1} = (sp)^{e - \varepsilon_2 - 1}s$. Thus, it follows that

$$(\bar{s}_1 \bar{p}_1)h = [(sp)^{e-\varepsilon_2-1}s]\,[(ps)^{\varepsilon_1}p] = (sp)^{e+\varepsilon_1-\varepsilon_2}.$$

Now, since both $\bar{x}$ and $\bar{s}_1 \bar{p}_1 \neq \lambda$ and h is nonerasing, their images under $h \neq \lambda$. Moreover, since h is 1–1, it follows that $\bar{x} = w^{\alpha_1}$ and $\bar{s}_1 \bar{p}_1 = w^{\alpha_2}$, where w is a primitive string (i.e. if $\bar{x}$ and $\bar{s}_1 \bar{p}_1$ were powers of two distinct primitive roots, it would follow that h is not 1–1). Thus, by our supposition, since $|\bar{x}|$ and $|\bar{s}_1 \bar{p}_1| < |y|$, it follows that $|w| \leqslant |sp| = |v|$.

But now it follows that $\bar{x}\bar{s}_1\bar{p}_1 = w^{\alpha_1 + \alpha_2}$. Thus, $\bar{p}_1 = \bar{s}_w w^{\bar{\varepsilon}}$, where $\bar{p}_w \bar{s}_w = w$, $\bar{p}_w, \bar{s}_w \in A^*$, $\bar{\varepsilon} \geqslant 0$ and since $\bar{x} = w^{\alpha_1}$, $\bar{s}_1 = w^{\alpha_2 - \bar{\varepsilon} - 1}\bar{p}_w$. Hence,

$$\bar{p}_1 \bar{x} \bar{s}_1 = (\bar{s}_w \bar{p}_w)^{\alpha_1 + \alpha_2} = y.$$

Thus, since $\bar{s}_w \bar{p}_w$ and y are both primitive, $\alpha_1 + \alpha_2 = 1$ and $y = \bar{s}_w \bar{p}_w$. Thus, $|y| = |\bar{s}_w \bar{p}_w| = |w| \leqslant |v|$ which is a contradiction, thus establishing the proof of the theorem. □

This theorem establishes a fundamental fact about what happens in a PWD0L system in which H is piecewise-injective and periodicity occurs, i.e. if $x \in Per_{(S,0)}$, then the lengths of all the primitive roots in the sequence remain the same. However, we still need that the partition $\mathbf{P}$ of A^* refine the partition induced by the power equivalence to yield the following result.

Corollary 2.12. *Let $S_w = (A, \mathbf{P}, H, w)$ be a PWD0L system in which H is piecewise injective $\mathbf{P}$ coarsens the partition induced by the power equivalence. Assume S_w is periodic with index 0 and let k denote the length of the unique primitive root of w. Then, the period for periodicity (and equivalently in this case power repetitiveness) is $\leqslant |A|^k$. Hence, in general, periodicity of index i for $i \geqslant 0$ is decidable for a PWD0L system S_w under the aforementioned hypotheses.*

Proof. Since S_w is periodic with index 0, there exists a $p \geqslant 1$ such that $(w)H^p = w^e$ for some $e \geqslant 1$. *Let* v denote the unique primitive root of w. Since v is primitive and $\mathbf{P}$ coarsens the partition induced by the power equivalence, it follows that $\Omega_{S_w} = \Omega_{S_v}$ and $(v)H^p = v^f$, where $f \geqslant 1$. Now, from Theorem 2.11, it follows that the lengths of all the primitive roots of the iterates of v under H must remain the same since S_v is periodic with index 0. Hence, from the pigeonhole principle, there exists a nonnegative integer i and positive integer j such that $i + j \leqslant |A|^k$ and $(v)H^i$ and $(v)H^{i+j}$ are powers of the same primitive root. Thus, since $\mathbf{P}$ coarsens the partition induced by the power equivalence, $(v)H^{i+dj}$ are powers of the same unique primitive root for every $d \geqslant 0$ and so from the pigeonhole principle, it follows that $p \leqslant |A|^k$ as required. □

Note that in [4], it is shown that if we also require the piecewise-endomorphism $H = (h_1, \ldots, h_k)$ to have the property that $|(a)h_i| \geqslant 2$ for each $i \in \{1, \ldots, k\}$ and $a \in A$, that $\{\text{primitives } v \mid (v)H^p = v^e \text{ for some } p \geqslant 1 \text{ and } e \geqslant 2\}$ is an algorithmically constructable finite set, and so Per_S and $Per_{(S,i)}$ for $i \geqslant 0$ are constructable regular sets over a finite recursive partition of A^*. Note that the hypotheses on the partition can be weakened so that only sufficiently high powers of words must be in the same class; but an increase in the period bound will occur in such cases. Next, an undecidability result for D0L systems will be stated which will allow us to find undecidability results for problems involving PWD0L systems S_w over a 2-element CS partition of A^*.

3. Undecidable properties of PWD0L systems

A generalized version of the following lemma is proven in [2]. This result is proven directly in [4].

Lemma 3.1. *Let A be a finite alphabet and let $L \subseteq A^*$ be context sensitive. Consider the following alphabet $\bar{A}$ which is the disjoint union of A and $\{V, \$\}$ and consider $S_L = \$Shuf(\bar{A}^*, L)\$ \subseteq \bar{A}^*$ where*

$$Shuf(\bar{A}^*, L) = \left\{ w \in \bar{A}^* \;\middle|\; w = \left(\prod_{i=1}^{k} p_i l_i\right) s, \text{ where } k \geqslant 0,\ l_i \in A,\ p_i, s \in \bar{A}^* \text{ and } \prod_{i=1}^{k} l_i \in L \right\}$$

Then, S_L is context sensitive.

Hence, using the previous lemma, the following undecidability result for D0L systems will be proven now.

Theorem 3.2. *There does not exist an algorithm that when given an arbitrary finite alphabet A, a D0L system $S_w = (A, h, w)$, where h is injective, and a context-sensitive language L, decides whether or not $L(S_w) \cap L = \emptyset$.*

To help prove this theorem, the following well-known result from classical formal language theory will be used.

Theorem 3.3 (Hopcroft and Ullman [6]). *There does not exist an algorithm that when given an arbitrary finite alphabet A ($|A| \geqslant 1$) and an arbitrary CS language L, decides whether or not $L = \emptyset$.*

Proof of Theorem 3.2. Suppose such an algorithm existed. Given a finite alphabet A ($|A| \geqslant 1$) and arbitrary CS language $L \subseteq A^*$, consider $\bar{A}$ and $S_L \subseteq \bar{A}^*$ as defined in the proof of Lemma 3.1 (see [4]). Finally, consider the following D0L system $S_w = (\bar{A}, h, w)$, where $w = \$V\$$ and define $h: \bar{A} \to \bar{A}^+$ by

$$(l)h = l \text{ if } l \in A \cup \{\$\} \quad \text{and} \quad (V)h = l_1 \dots l_{|A|} V,$$
where l_i is the ith element of A $(1 \leqslant i \leqslant |A|)$.

Then, I claim that it follows by induction that $(w)h^k = \$(l_1 \dots l_{|A|})^k V\$$ for all $k \geqslant 0$. Moreover, it follows that any string in A^k is a finite subsequence of $(w)h^k$. Since we defined S_L to be the same as in the statement of Lemma 3.1 (hence S_L is CS), if follows that $L = \emptyset$ if and only if $L(S_w) \cap S_L = \emptyset$. Thus, if we had an algorithm to decide what is desired in the statement of this theorem, the emptiness problem for CS languages could be decided which is a contradiction. Hence, the theorem is established. □

Lemma 3.1. and Theorem 3.2 will allow a battery of undecidability results for PWD0L systems over two-element CS partitions of A^* to be proven now.

Corollary 3.4. *Let A be an arbitrary finite alphabet, $H = (h, 1_{A*})$ be a piecewise-injective piecewise endomorphism over $\mathbf{P} = \{L^c, L\}$, where L is context sensitive. Then, for a PWD0L system of the form $S_w = (A, \mathbf{P}, H, w)$, the following problems are, in general, undecidable:*
(i) *finiteness,*
(ii) *periodicity,*
i.e. given an arbitrary PWD0L system of the form given above, there does not exist an algorithm which will decide these problems in general.

Proof. Suppose such an algorithm existed. Let L be an arbitrary CS language over A^* and consider the alphabet $\bar{A}$, language $S_L \subseteq \bar{A}^*$ and word $\$V\$$ which were considered in Lemma 3.1 and Theorem 3.2, respectively. Consider the following PWD0L system:

$$\bar{S}_w = (\bar{A}, \mathbf{P}, H, w),$$

where $\mathbf{P} = \{S_{L^c}, S_L\}$ and $H = (h, 1_{\bar{A}^*})$, where h is the injective morphism defined in the proof of Theorem 3.2. I claim that the equivalence of the following four properties follows easily from Theorem 3.2:
(1) $L(\bar{S}_w)$ is finite.
(2) $\bar{S}_w$ is periodic.
(3) $L(\bar{S}_w) \cap S_L \neq \emptyset$.
(4) $L \neq \emptyset$.
Thus, if we had an algorithm of the form stated in this corollary, we could again, in general, decide the emptiness problem of CS languages, which is contradictory. Thus, the corollary is established. □

By making very slight adjustments to the type of PWD0L system talked about in Corollary 3.4, many more undecidability results can be stated. Corollaries 3.5 and 3.6 will deal with this.

Corollary 3.5. *Let A be a finite alphabet and $S_w = (A, \mathbf{P}, H, w)$ be a PWD0L system, where $\mathbf{P} = \{L^c, L\}$ (L is CS), $H = (h_1, h_2)$, where h_1 is injective. Then, for a PWD0L system of the aforementioned form, the following problems are, in general, undecidable:*
(i) *finiteness with index* 0,
(ii) *periodicity with index* 0,
(iii) $\lambda \in L(S_w)$.

Proof. Use a proof similar to the proof of Corollary 3.4 except that you make the following modifications:
(I) $\bar{\bar{A}} = \bar{A} \cup \{\bar{\$}\}$,
(II) $S_w = (\bar{\bar{A}}, \mathbf{P}, H, w)$,
where $w = \$V\$$ and $H = (h_1, h_2)$, where h_1 is the same as h in the proof of theorem 3.2 (with the addition that $(\bar{\$})h = \bar{\$}$). The language S_L is the same as the definition in Lemma 3.1 except that words must end with $\bar{\$}$ rather than \$. To get undecidability results for (i) and (ii), define $h_2 : \bar{\bar{A}} \to \bar{\bar{A}}^*$ by $(l)h_2 = \lambda$ if $l \neq \$$ and $(\$)h_2 = w$. Finally, to get undecidability for (iii), define $h_2 = \Lambda$, i.e. the endomorphism that takes every letter in $\bar{\bar{A}}$ to λ. In both cases, it follows that $L(S_w)$ has the required property if and only if $L \neq \emptyset$, where L is an arbitrary CS language over A^*. Thus, (i)–(iii) are, in general, undecidable. □

Corollary 3.6. *For a PWD0L systems of the form $S_w = (A, \mathbf{P}, H, w)$, where A and $\mathbf{P}$ are ,as in the statement of Corollary 3.5, and $H = (h_1, h_2)$, where h_1 is injective and h_2 is λ-free, power repetitiveness with index 0 is, in general, undecidable. Finally, if $H = (h_1, h_2)$, where both h_1 and h_2 are injective, then nonprefixness and nonprefixness with index 0 are, in general, undecidable.*

Idea of proof. Make slight modifications to the systems used in the proofs of Corollaries 3.4 and 3.5. □

In Section 4, the relationship between morphic equivalence relations and PWD0L systems will be examined.

4. Partition-preserving piecewise endomorphism and PWD0L systems

We will first define the notion of a piecewise endomorphism H of A^* preserving a partition $\mathbf{P}$ of A^*.

Definition 4.1. Let $\mathbf{P} = \{P_1, \ldots, P_k\}$ be a finite partition of A^* induced by an equivalence relation $\sim$. Let $H = (h_1, \ldots, h_k)$ denote a piecewise endomorphism of A^* (given $\mathbf{P}$). Then, H preserves $\mathbf{P}$ if for every $i \in \{1, \ldots, k\}$, there exists a $j \in \{1, \ldots, k\}$ such that $(P_i)H \subseteq P_j$.

Note that given a finite partition of all regular languages, it is in general decidable if H preserves this partition. However, given an arbitrary two-element context-free partition of A^* and a piecewise endomorphism over $\mathbf{P}$, this problem is, in general, undecidable (see [4]).

Given that H preserves $\mathbf{P}$, many dynamical properties of PWD0L systems can be reduced to deciding an analogous property for an alternative D0L system which will be shown below.

Proposition 4.2. *Let $\mathbf{P} = \{P_1, \ldots, P_k\}$ be a finite partition of A^*, $H = (h_1, \ldots, h_k)$ be a piecewise endomorphism over $\mathbf{P}$ such that H preserves $\mathbf{P}$. Let $S = (A, \mathbf{P}, H)$ denote the corresponding PWD0L scheme. Then, for each $i \in \{1, \ldots, k\}$, there exists $u_i \in \{1, \ldots, k\}^*$ and $v_i \in \{1, \ldots, k\}^+$ such that for every $x, y \in P_i$, the following properties hold:*
(1) $\Omega_{S_x} = \Omega_{S_y}$,
(2) $\Omega_{S_x} = u_i v_i^w$, where $|u_i| \leqslant k-1$, $1 \leqslant |v_i| \leqslant k$, u_i and v_i contain no repeated symbols, and no letter that appears in u_i appears in v_i.

Hence, for a PWD0L scheme S in which H preserves $\mathbf{P}$, it makes sense to talk about Ω_i, i.e. *the omega word generated by S with respect to P_i ($1 \leqslant i \leqslant k$).*

Proof. Let $i \in \{1, \ldots, k\}$. Since H preserves $\mathbf{P}$, it follows that for every $x, y \in P_i$, $\Omega_{S_x} = \Omega_{S_y}$. Thus, (1) is proven and, moreover, it makes sense to talk about the omega word generated by S with respect to P_i. It now suffices to prove (2).

Let α denote the smallest nonnegative integer such that for some $\bar{\beta} \geqslant 1$, $(x)H^{\alpha}$ and $(x)H^{\alpha+\bar{\beta}}$ lie in the same member of $\mathbf{P}$. Given α, let β denote the smallest positive integer such that $(x)H^{\alpha} = (x)H^{\alpha+\beta}$. It follows from the pigeonhole principle that $\alpha \leqslant k-1$ and $1 \leqslant \beta \leqslant k$. Let u_i denote the prefix of length α of Ω_{S_x} and let v_i denote the subword of Ω_{S_x} of length β between positions $\alpha + 1$ and $\alpha + \beta$. Since H preserves $\mathbf{P}$, it follows that $\Omega_{S_x} = u_i v_i^w$, where u_i and v_i contain no repeated symbols or common symbols as required. □

This proposition gives us the machinery to help prove the following important and fundamental proposition that shows how if H preserves $\mathbf{P}$, finiteness and periodicity for a PWD0L system over $\mathbf{P}$ can be decided relatively easily.

Proposition 4.3. *Let $S_w = (A, \mathbf{P}, H, w)$ be a PWD0L system over a finite partition $\mathbf{P}$ of A^*. Then*

(1) S_w is finite if and only if T_w is finite, where $T_w = (A, h_v, (w)h_u)$ is a D0L system, where $\Omega_{S_w} = uv^w$. Note that periodic or BOUNDED can be substituted for finite in the previous statement.

(2) S_w is finite with index 0 if and only if $\Omega_{S_w} = v^w$ and $T_w = (A, h_v, w)$ is finite with index 0. Note that periodic with index 0 can be substituted for finite with index 0 in the previous statement if $\mathbf{P}$ coarsens the power equivalence.

This proposition will be proved in the periodic case.

Proof. ($\Leftarrow$) since T_w is periodic, it follows that S_w is periodic with index $\leqslant |u|$.

($\Rightarrow$) Assume S_w is periodic with some index $i \geqslant 0$. From Proposition 4.2, we know that $\Omega_{S_w} = uv^w$ is ultimately periodic. Thus, from the pigeonhole principle and the fact that each h_i is an endomorphism for $i \in \{1, \ldots, k\}$, it follows that there exists $0 < \varepsilon_1 < \varepsilon_2$ such that

$$(w)h_{uv^{\varepsilon_1}p} = ((w)h_\kappa)^{e_1}$$

and

$$(w)h_{uv^{\varepsilon_2}p} = (((w)h_\kappa)^{e_1})h_{(sp)^{\varepsilon_2-\varepsilon_1}} = ((w)h_\kappa)^{e_1 e_2} = ((w)h_{uv^{\varepsilon_1}p})^{e_2},$$

where κ is the prefix of Ω_{S_w} of length $i, e_1, e_2 \geqslant 0$ and $ps = v$. Thus, since h_s is an endomorphism and Ω_{S_w} is ultimately periodic, it follows that $(w)h_{uv^{\varepsilon_2}+1} = (w^{e_2})h_{uv^{\varepsilon_1}+1} = ((w)h_{uv^{\varepsilon_1}+1})^{e_2}$ and so T_w is periodic as required.

To obtain the claimed result in the finite with index 0 case or periodic with index 0 case (assuming **P** coarsens the partition induced by the power equivalence), it is essential to observe that $\Omega_{S_w} = v^w$, where $|v| \leqslant k$, i.e. $u = \lambda$. □

Hence, we can decide certain properties of PWD0L languages if the piecewise endomorphism H preserves **P**. However, using the information provided in Theorem 2.4, the following two corollaries will allow strengthening of the previous result.

Corollary 4.4. *Let $S = (A, \mathbf{P}, H)$ be a PWD0L scheme such that H preserves $\mathbf{P} = \{P_1, \ldots, P_k\}$. Then*
(1) $Fin_S = \bigcup_{j=1}^{k} ((Fin_{S_j})h_{u_j}^{-1} \cap P_j)$ and
(2) $Fin_{(S,0)} = \bigcup_{j \in \{1,\ldots,k\}, \Omega_j = v_j^\omega} ((Fin_{(S_j,0)}) \cap P_j)$,
*where $S_j = (A, h_{v_j})$ is a D0L scheme for $j \in \{1, \ldots, k\}$ and $\Omega_j = u_j v_j^w$ is the omega word generated by S for any $x \in P_j$. Note that Per_S or $BOUNDED_S$ can be substituted for Fin_S in (1) and $Per_{(S,0)}$ can be substituted for $Fin_{(S,0)}$ in (2) if **P** coarsens the partition of A^* induced by the power equivalence.*

Proof. First (1) will be proven for Per_S and then (2) for $Fin_{(S,0)}$.

(1) Assume $x \in Per_S \cap P_j$ for some $j \in \{1, \ldots, k\}$. Then, S_x is periodic. Let $\Omega_j = \Omega_{S_x} = u_j v_j^w$ be the omega word generated by S_x. Then, by Proposition 4.3, S_x is periodic if and only if $S_{j_x} = (A, h_{v_j}, (x)h_{u_j})$ is periodic. Thus, $x \in (Per_{S_j})h_{u_j}^{-1}$ and so $x \in (Per_{S_j})h_{u_j}^{-1} \cap P_j$ for some j as required.

Conversely, assume $x \in (Per_{S_j})h_{u_j}^{-1} \cap P_j$ for some j, where $S_j = (A, h_{v_j})$ is a D0L scheme and $\Omega_{S_x} = \Omega_j = u_j v_j^w$. Thus, $(A, h_{v_j}, (x)h_{u_j})$ is periodic and thus by Proposition 4.3, S_x is periodic. Thus, x is in Per_S as required.

(2) It follows from Proposition 4.3 that $x \in Fin_{(S,0)} \Rightarrow$ the control word generated by S_x must he periodic. Moreover, $(x)H^{k_j} = x$, where $k_j = \alpha|v_j|$, where $\alpha \geqslant 1$. Hence, it suffices to consider only the members of **P** whose control words are periodic. Thus, using an almost identical proof to (1), the result follows. □

Corollary 4.5. *Let $S = (A, \mathbf{P}, H)$ be a PWD0L scheme in which H preserves $\mathbf{P}$. If each member of $\mathbf{P}$ is regular (CF, CS, recursive), then Fin_S, Per_S, $BOUNDED_S$, and $Fin_{(S,i)}$ are regular (CS, CS, recursive) languages for $i \geqslant 0$. If $\mathbf{P}$ coarsens the power equivalence, then $Per_{(S,i)}$ is regular (CS, CS, recursive) for $i \geqslant 0$.*

Idea of proof. Follows from the closure properties of regular (CF, CS, recursive) languages, Corollary 4.4 and Theorem 2.4. □

It also follows that if we have a finite state representation of each member of the partition $\mathbf{P}$ (for example, if each P_i is regular, a finite automaton which accepts each P_i), then any of the sets of strings mentioned in the conclusion of Corollary 4.5 can be effectively constructed. Also note that in the proof of Proposition 4.3, the proof only depended upon the fact that $\Omega_{S_x} = uv^w$ is ultimately periodic.

In Section 5 of this paper, a boolean algebra **Morph** of languages will be presented in which many properties, such as the regularity of Fin_S and Per_S for a PWD0L scheme S holds for finite partitions in which every element is from **Morph**.

5. The classes Morph and Morph$^{\mathrm{cong}}$

First, the definition of a morphic equivalence on A^* and two classes of languages will be presented.

Definition 5.1. Let $\sim$ be an equivalence relation on A^*. Then $\sim$ is a *morphic equivalence* on A^* if for every endomorphism $h: A \to A^*$, $x \sim y$ implies that $(x)h \sim (y)h$. If $\mathbf{M}$ is a partition of A^* induced by a morphic equivalence, then $\mathbf{M}$ is called a *morphic partition* of A^*.

Definition 5.2. Let A be a finite alphabet. Define

$$\mathbf{Morph} = \{L \subseteq A^* \mid \mathbf{P} = \{L, L^c\} \text{ is refined by a finite recursive morphic partition } \mathbf{M} \text{ of } A^*\} \text{ and}$$

$$\mathbf{Morph}^{\mathrm{cong}} = \{L \subseteq A^* \mid \mathbf{P} = \{L, L^c\} \text{ is refined by a finite morphic partition } \mathbf{M} \text{ of } A^* \text{ induced by a morphic congruence relation } \sim\}.$$

Note that **Morph**$^{\mathrm{cong}}$ is a subset of the class of all regular languages. The first property that can be easily seen about **Morph** is that it is a boolean algebra with respect to set union, intersection and complementation. For a proof, see [3].

Many dynamical properties of PWD0L systems (such as finiteness and periodicity) over finite partitions of A^* in which every member of the partition is from **Morph** can be decided in general if a common morphic refinement of this partition can be effectively constructed. This will be proven in the following three corollaries.

Corollary 5.3. *Assume that* $\mathbf{P} = \{P_1, \dots, P_k\}$ *is a paritition of* A^* *in which every element* $P_i \in \mathbf{Morph}$, $1 \leqslant i \leqslant k$. *Let* $S = (A, \mathbf{P}, H)$ *denote a PWD0L scheme over* $\mathbf{P}$. *Then, for* $j \geqslant 0$, Fin_S, Per_S *and* $Fin_{(S,j)}$ *are regular (CS, CS, recursive) subsets of* A^* *given that each member of the morphic refinement of* $\mathbf{P}$ *is regular (CF, CS, recursive).*

Sketch of proof. Since each P_i is in **Morph**, there exists a finite morphic refinement $\mathbf{M}_i$ of $\{P_i, P_i^c\}$. Consider the partition of A^* with classes $\bigcap_{i=1}^k M_{ij_i}$, where $M_{ij_i} \in \mathbf{M}_i$ for each $i \in \{1, \dots, k\}$. Then, it follows that this partition is a finite morphic refinement of $\{P_1, \dots, P_k\}$. Now, by applying the original piecewise endomorphism to this refinement, and using Corollary 4.5, the result follows. □

We will now discuss a characterization of the regular languages which will allow us to obtain some decidability results for RWD0L systems.

The following theorem is analogous to a theorem of Birkhoff [1] involving a finitely generated free algebra in the variety generated by a finite algebra. A proof is given in either [3] or [4].

Theorem 5.4 (Harrison [3] and [4]). *Let* $\sim$ *be a congruence of finite index on* A^*, *where* A *is a finite alphabet. Then, there exists a morphic refinement* $\sim_{\mathrm{M}}$ *of* $\sim$ *of finite index. Thus,* $\mathbf{Morph}^{\mathrm{cong}} = \mathbf{Reg}$.

Moreover, given a regular language L and its complement, it is shown in [3, 4] that such a refinement can be algorithmically constructed by algorithmically passing to the syntactic refinement of L. Hence, the following corollaries are obtained for RWD0L schemes and systems.

Corollary 5.5. *Let* S *be an arbitrary RWD0L scheme. Then* Fin_S, Per_S, $BOUNDED_S$, *and* $Fin_{(S,j)}$ *are algorithmically constructible regular sets for each* $i \geqslant 0$. *Finally, if the underlying morphic refinement* $\mathbf{M}$ *coarsens the power equivalence, then* $Per_{(S,i)}$ *is also an algorithmically constructable regular set for all* $i \geqslant 0$.

Proof. Follows directly from Corollary 4.5 and Theorem 5.4. □

We also obtain the following corollary given that any D0L language is context sensitive (see [5]) and Theorem 5.4.

Corollary 5.6. *Let* L *be an RWD0L language. Then, there exists* $k \geqslant 0$ *such that* $L = F \cup \bigcup_{i=1}^k L_i$, *where* F *is a finite language and each* L_i *is a D0L language. Thus,* L *is context sensitive.*

Proof. From Theorem 5.4, if follows that $\mathbf{P}$ has a morphic refinement of finite index. Let $S = (A, \mathbf{P}, H, w)$ be a PWD0L system such that $L = L(S_w)$. Since $\mathbf{P}$ has a finite morphic refinement, it follows from Proposition 4.2 that $\Omega_{S_w} = uv^w$, where $|u|, |v|$ are both $\leqslant$ to the number of classes in the morphic refinement of **P**. Hence, the desired finite set is $F = \bigcup_{p \leqslant u} \{(w)h_p\}$. Finally, for each proper prefix $\bar{p}$ of v, consider $L_{\bar{p}} = L(T_{(w)hu_{\bar{p}}})$, where $(T_{(w)hu_{\bar{p}}}) = (A, h_{\bar{s}\bar{p}}, (w)h_{u\bar{p}})$ is a D0L system for each $\bar{p}$ (where $\bar{p}\bar{s} = v$). Then, $L = F \cup \bigcup_{\bar{p} < v} L_{\bar{p}}$. Finally, the fact that L is context sensitive follows from the proof in [5] and the closure properties of the CS languages [6]. □

6. Summary and related work

This is the first paper done on PWD0L systems, and it is my hope that other researchers will become interested in this area. It is also my hope that PWD0L systems may become as widely applied as D0L systems as a development model for plant development. Morphic congruences have been applied to studying D0L systems in [3]. For more information on PWD0L systems, see [4].

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References

[1] G. Birkhoff, *Lattice Theory* (Amer. Math. Soc., Providence, RI, 3rd ed., 1967).
[2] S. Ginsberg, *The Mathematical Theory of Context-Free Languages* (McGraw-Hill, New York, 1966).
[3] J. Harrison, Morphic congruences and D0L languages, *Theoret. Comput. Sci.* **134** (1994)537–544.
[4] J. Harrison, The dynamics of piecewise endomorphisms of finitely-generated free monoids, Ph.D. Thesis, Department of Mathematical Sciences, Binghamton University (SUNY), 1993.
[5] G.T. Herman and G. Rozenberg, *Developmental Systems and Languages* (North-Holland, Amsterdam, 1975).
[6] J. Hopcroft and J. Ullman, *Introduction to Automata Theory, Languages and Computation* (Addison-Wesley, Reading, MA, 1979).
[7] N. Immerman, Nondeterministic space is closed under complementation, *SIAM J. Comput.* **17** (1988) 935–938.
[8] B. Lando, Periodicity and ultimate periodicity of D0L systems, *Theoret. Comput. Sci.* **82** (1991) 19–33.
[9] M. Linna, The decidability of the D0L prefix problem, *Internat. J. Comput. Math.* **6** (1977).
[10] M. Lothaire, *Combinatorics on Words* (Addison-Wesley, Reading, MA, 1983).
[11] J.E. Pin, *Varieties of Formal Languages* (Plenum Press, New York, 1986).